NORTHWESTERN UNIVERSITY
TRAINING/SELECTION TEST

Thursday, October 6, 2005

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WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

Problem A1. Find the sum $\sum_{k=0}^{n}(3k(k+1)+1)$, for $n \geq 1$.

- *Answer:*

The simplest way to solve this problem is to use the fact that the terms of the sum are differences of consecutive cubes:

$$3k(k+1)+1 = 3k^2+3k+1 = (k+1)^3 - k^3\,,$$

so the sum telescopes:

$$\sum_{k=0}^{n}(3k(k+1)+1) = (1^3-0^3)+(2^3-1^3)+\cdots+((n+1)^3-n^3) = (n+1)^3\,.$$

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Problem A2. Given a fix positive integer n, find the minimum value of the following function:

$$f(x) = x^n + x^{n-2} + x^{n-4} + \cdots + \frac{1}{x^{n-4}} + \frac{1}{x^{n-2}} + \frac{1}{x^n}$$

for $x > 0$.

- *Answer:*

We will prove that the minimum is $n+1$. To that end, we group the terms in the given expression as follows:

$$f(x) = \left(x^n + \frac{1}{x^n}\right) + \left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots$$

The last term may be 1 or $x + 1/x$ depending on the parity of n.

Next we use the inequality $y + 1/y \geq 2$ for $y > 0$, which is easily derived from $(y-1)^2 \geq 0$. So, each term between parenthesis is bounded below by 2. If n is odd there are $(n+1)/2$ of those terms. If n is even there are $n/2$ terms of the form $x^k + 1/x^k$, plus an extra term equal to 1. In any case we get $f(x) \geq n+1$. Finally we notice that the lower bound is attained at x=1.

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Problem A3. On a large, flat field, n people ($n > 1$) are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry.

- *Answer:*

We use induction.

First note that among all distances between any two people one must be minimum, and that two people placed a that minimum distance must shoot each other. For $n = 3$ (base case) the third person will obviously stay dry, so this proves the statement for $n = 3$.

For the induction step, we assume the statement to be true for a given odd $n > 1$, and then we must prove it for $n + 2$. As before, two people, placed at minimum distance must shoot each other. Call those two people A and B. Among the remaining n people, if nobody shoots A or B, we can apply the induction hypothesis to conclude that someone among them will be left dry. Otherwise, if someone shoots A or B, we will have a group of n people receiving less than n shots of water. By the Pigeonhole Principle, one of them will stay dray.

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Problem A4. $\mathbf{R}$ is the set of real numbers. For what $k \in \mathbf{R}$ can we find a continuous function $f : \mathbf{R} \to \mathbf{R}$ such that

$$f(f(x)) = kx^9$$

for all $x \in \mathbf{R}$.

- *Answer:*

The answer is for any $k \geq 0$.

If $k \geq 0$, one function with the required property is $f(x) = k^{1/4}\, x^3$.

Next we must prove that there is no such function if $k < 0$.

First note that for $k \neq 0$, the function $x \mapsto f(f(x)) = kx^9$ is a bijection (or a 1-to-1 correspondence) from $\mathbf{R}$ to $\mathbf{R}$. Consequently f itself is a (continuous) bijection from $\mathbf{R}$ to $\mathbf{R}$. A continuous bijection from $\mathbf{R}$ to $\mathbf{R}$ must be monotonous—either increasing or decreasing. But whether f is increasing or decreasing, $x \mapsto f(f(x))$ will be increasing, so k must be positive.

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Problem A5. Show that for any positive integer n, there exists a positive multiple of n that contains only the digits 7 and 0.

- *Answer:*

Method 1: This can be solved by the Pigeonhole Principle. Look at the infinite sequence $a_1 = 7$, $a_2 = 77$, $a_3 = 777$, $a_4 = 7777, \dots, a_k = 7(10^k - 1)/9, \dots$ Then look at the sequence of residues modulo n of its elements. Since there are n residues modulo n, eventually two of them will coincide: $a_r \equiv a_s \pmod{n}$, $0 < r < s$. Then $a_s - a_r$ will be a multiple of n containing only the digits 7 and 0 (in fact consisting of a string of 7's followed by a string of 0's.)

Method 2: We proceed as above, but using instead the sequence $a_0 = 7$, $a_1 = 70$, $a_2 = 700$, $a_3 = 7000, \dots, a_k = 7 \cdot 10^k, \dots$ The first n^2 terms will be distributed among n residual classes modulo n, so in some class there must be at least n terms: $a_{k_i} \equiv a \pmod{n}$, $i = 1, 2, \dots, n$. their sum verifies $a_{k_1} + a_{k_2} + \cdots + a_{k_n} \equiv na \equiv 0 \pmod{n}$, so it is a multiple of n. And obviously contains only the digits 7 and 0, as required.

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Problem A6. Let u_n be the number of symmetric $n \times n$-matrices whose elements are all 0's and 1's with exactly one 1 in each row. Let $u_0 = 1$. Prove that

$$u_{n+1} = u_n + n u_{n-1}$$

and

$$\sum_{n=0}^{\infty} u_n \frac{x^n}{n!} = e^{x+x^2/2}.$$

- *Answer:*

The recurrence relation can be obtained as follows. For $n = 0$ and $n = 1$ it can be checked directly. If $n \geq 2$, the $(n+1) \times (n+1)$ matrices of the kind indicated above can be divided into two classes, depending on the column position of the "1" in their last row:

1. The "1" in their last row is in the last column.
2. The "1" in their last row is in column k, with $1 \leq k \leq n$.

In case 1, if we eliminate the last row and column, we get an $n \times n$ matrix as indicated above. There are u_n such matrices. In case 2, if we remove the last row and row and also the kth row and column, we get an $(n-1) \times (n-1)$ matrix as indicated above. There are u_{n-1} such matrices. And k can have n possible values.

Once we have established the recurrence, we can justify the equation by differentiating both sides and checking that they satisfy the same differential equation:

$$f'(x) = (1 + x) f(x)$$

with the same initial condition $f(0) = 1$.