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# FALL 2011 NU PUTNAM SELECTION TEST

**Problem A1.** Let $a_1, a_2, \ldots, a_n$ be $n$ not necessarily distinct integers. Prove that there exist a subset of these numbers whose sum is divisible by $n$.

- *Answer:* Consider the numbers $s_1 = a_1$, $s_2 = a_1 + a_2$,..., $s_n = a_1 + a_2 + \cdots + a_n$. If any of them is divisible by $n$ then we are done, otherwise their remainders are different from zero modulo $n$. Since there are only $n - 1$ such remainders, two of the sums, say $s_p$ and $s_q$, with $p < q$, will have the same reminder modulo $n$, and $s_q - s_p = a_{p+1} + \cdots + a_q$ will be divisible by $n$.

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**Problem A2.** If $a$, $b$, and $c$ are the sides of a triangle, prove that

$$\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}\geq 3\,.$$

- *Answer:*

- Solution 1:

Step 0 - The idea is that the lower bound 3 is attained when three sides are equal.

Step 1 - Replace both $a$ and $b$ by $(a+b)/2$, you can show by algebra that the sum is reduced.

Step 2 - When $a$ and $b$ are equal, the inequality, you can check easily that the inequality holds.

- Solution 2: Set $x=b+c-a$, $y=c+a-b$, $z=a+b-c$. The triangle inequality implies that $x$, $y$, and $z$ are positive. Furthermore, $a=(y+z)/2$, $b=(z+x)/2$, and $c=(x+y)/2$. The LHS of the inequality becomes:

$$\frac{y+z}{2x}+\frac{z+x}{2y}+\frac{x+y}{2z}=\frac{1}{2}\left(\frac{x}{y}+\frac{y}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{z}+\frac{z}{x}\right)\geq 3\,.$$

- Solution 3: Work like in solution 2, but write the LHS like this:

$$\frac{y+z}{2x}+\frac{z+x}{2y}+\frac{x+y}{2z}=\frac{x+y+z}{2x}+\frac{x+y+z}{2y}+\frac{x+y+z}{2z}-\frac{3}{2}\,.$$

By the AM-GM inequality we have that expression is greater than or equal to

$$3\frac{3}{\frac{2x}{x+y+z}+\frac{2y}{x+y+z}+\frac{2z}{x+y+z}}-\frac{3}{2}=3\,,$$

and we are done.

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**Problem A3.** Does there exist a positive sequence $a_n$ such that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} 1/(n^2 a_n)$ are convergent?

- *Answer:* There is no such sequence. If they were convergent their sum would be convergent too, but by the AM-GM inequality we have:

$$\sum_{n=1}^{\infty} \left( a_n + \frac{1}{n^2 a_n} \right) \geq \sum_{n=1}^{\infty} \frac{2}{n} = \infty \, .$$

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## FALL 2011 NU PUTNAM SELECTION TEST

**Problem A4.** On a table there is a row of fifty coins, of various denominations (the denominations could be of any values). Alice picks a coin from one of the ends and puts it in her pocket, then Bob chooses a coin from one of the ends and puts it in his pocket, and the alternation continues until Bob pockets the last coin. Prove that Alice can play so that she guarantees at least as much money as Bob.

- *Answer:* Alice adds the values of the coins in odd positions 1st, 3rd, 5th, etc., getting a sum $S_{odd}$. Then she does the same with the coins placed in even positions 2nd, 4th, 6th, etc., and gets a sum $S_{even}$. Assume that $S_{odd} \geq S_{even}$. Then she will pick all the coins in odd positions, forcing Bob to pick only coins in the even positions. To do so she stars by picking the coin in position 1, so Bob can pick only the coins in position 2 or 50. If he picks the coin in position 2, Alice will the pick coin in position 3, if he picks the coin in position 50 she picks the coin in position 49, and so on, with Alice always picking the coin at the same side as the coin picked by Bob.

If $S_{odd} \leq S_{even}$, then Alice will use a similar strategy ensuring that she will end up picking all the coins in the even positions, and forcing Bob to pick the coins in the odd positions—this time she will pick first the 50th coin, and then at each step she will pick a coin at the same side as the coin picked by Bob.

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## FALL 2011 NU PUTNAM SELECTION TEST

**Problem A5.** Prove that there is no polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients and of degree at least 1 with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

- *Answer:*

- Solution 1: By contradiction, assume that $P(k)$ is a prime number for every $k = 0, 1, 2, \ldots$. We have

$$P(P(k) + k) \equiv P(k) \equiv 0 \pmod{P(k)},$$

Hence $P(P(k) + k)$ is divisible by $P(k)$. Since $P(P(k) + k)$ and $P(k)$ are both prime by hypothesis, we have $P(P(k)+k) = P(k)$ for every $k \geq 0$, hence $P(P(x)+x)$ and $P(x)$ must be the same polynomial. If $P(x)$ is of degree $d$, then $P(P(x)+x)$ has degree $d^2$, and from here we get that the only possibility is $d = 1$. If $P(x) = ax+b$, then $P(P(x)+x) = (a^2+a)x+(ab+b)$, and if both are the same polynomial that implies $a^2+a = a$, hence $a = 0$, which is impossible.

- Solution 2: Also by contradiction. We have that $a_0 = P(0)$ must be a prime number. Also, $P(ka_0)$ is a multiple of $a_0$ for every $k = 0, 1, 2, \ldots$, but if $P(ka_0)$ is prime then $P(ka_0) = a_0$ for every $k \geq 0$. This implies that the polynomial $Q(x) = P(a_0x) - a_0$ has infinitely many roots, so it is identically zero, and $P(a_0x) = a_0$, contradicting the hypothesis that $P$ is of degree at least 1.

$\square$

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## FALL 2011 NU PUTNAM SELECTION TEST

**Problem A6.** Given thirteen real numbers $r_1, r_2, \ldots, r_{13}$, prove that there are two of them $r_p$, $r_q$, $p \neq q$, such that $|r_p - r_q| \leq (2-\sqrt{3})|1 + r_p r_q|$. (Note: $2 - \sqrt{3} = \tan \frac{\pi}{12}$.)

- *Answer:* For each $i = 1, \ldots 13$, let $\alpha_i$ be the angle in the interval $(-\pi/2, \pi/2)$ such that $r_i = \tan \alpha_i$. Then two of those angles are at a distance not greater than than $\pi/12$, say $|\alpha_p - \alpha_q| \leq \pi/12$. Hence

$$2 - \sqrt{3} = \tan \tfrac{\pi}{12} \geq |\tan(\alpha_p - \alpha_q)| = \frac{|\tan(\alpha_p) - \tan(\alpha_q)|}{|1 + \tan(\alpha_p)\tan(\alpha_q)|} = \frac{|r_p - r_q|}{|1 + r_p r_q|},$$

and the result follows.

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## FALL 2011 NU PUTNAM SELECTION TEST

**Problem A7.** (Note: this question was not included in the final version of the test.) The digital root of a number is the (single digit) value obtained by repeatedly adding the (base 10) digits of the number, then the digits of the sum, and so on until obtaining a single digit—e.g. the digital root of 65,536 is 7, because $6+5+5+3+6 = 25$ and $2+5 = 7$. Consider the sequence $a_n$ = integer part of $10^n\pi$, i.e.,

$$a_1 = 31\,,\quad a_2 = 314\,,\quad a_3 = 3141\,,\quad a_4 = 31415\,,\quad a_5 = 314159\,,\quad \dots$$

and let $b_n$ be the sequence

$$b_1 = a_1\,,\quad b_2 = a_1^{a_2}\,,\quad b_3 = a_1^{a_2^{a_3}}\,,\quad b_4 = a_1^{a_2^{a_3^{a_4}}}\,,\quad \dots$$

Find the digital root of $b_{10^6}$.

- *Answer:* The problem may look hard, but it is very easy, because the digital root of $b_n$ becomes a constant very quickly. First note that the digital root of a number $a$ is just the reminder $r$ of $a$ modulo 9, and the digital root of $a^n$ will be the remainder of $r^n$ modulo 9.

For $a_1 = 31$ we have

digital root of $a_1$ = digital root of $31 = 4$ ;

digital root of $a_1^2$ = digital root of $4^2 = 7$;

digital root of $a_1^3$ = digital root of $4^3 = 1$;

digital root of $a_1^4$ = digital root of $4^4 = 4$;

and from here on it repeats with period 3, so the digital root of $a_1^n$ is 1, 4, and 7 for remainder modulo 3 of $n$ equal to 0, 1, and 2 respectively.

Next, we have $a_2 = 314 \equiv 2 \pmod 3$, $a_2^2 \equiv 2^2 \equiv 1 \pmod 3$, $a_2^3 \equiv 2^3 \equiv 2 \pmod 3$, and repeating with period 2, so the reminder of $a_2^n$ depends only on the parity of $n$, with $a_2^n \equiv 1 \pmod 3$ if $n$ is even, and $a_2^n \equiv 2 \pmod 3$ if $n$ is odd.

And we are done because $a_3$ is odd, and the exponent of $a_2$ in the power tower defining $b_n$ for every $n \geq 3$ is odd, so the reminder modulo 3 of the exponent of $a_1$ will be 2, and the reminder modulo 9 of $b_n$ will be 7 for every $n \geq 3$.

Hence, the answer is 7.

$\square$