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Problem A1. Find all integer solutions to the system of equations

$$\begin{cases} x^2 - y^2 = 16 \\ x^3 - y^3 = 98 \end{cases}$$

- *Answer:* We have $x^2 - y^2 = (x-y)(x+y)$, and $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ hence $x-y$ is a common factor of 16 and 98, i.e., a divisor of $\gcd(16,98) = 2$. The second equation $x^3 - y^3 = 98$ shows that $x > y$, hence $x - y > 0$, so the only possibilities are the positive divisors of 2, i.e., $x - y = 1$ and $x - y = 2$.

If $x - y = 1$, then $x + y = 16$, and from here we get $x = 17/2$, $y = 15/2$, which are not integers and do not satisfy $x^3 - y^3 = 98$.

If $x - y = 2$, then $x + y = 8$, and from here we get $x = 5$, $y = 3$, which do satisfy the system. So, the only integer solution is

$$\boxed{(x,y) = (5,3)}\,.$$

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Problem A2. Prove that 48 divides $n^4 - 1$ if n is not a multiple of 2 or 3.

- *Answer:* Since $48 = 16 \cdot 3$, it suffices to prove that (with the given conditions) 3 and 16 both divide $n^4 - 1$.

If n is not a multiple of 3, then $n \equiv \pm 1 \pmod 3$, and $n^4 \equiv 1 \pmod 3$, hence $n^4 - 1 \equiv 0 \pmod 3$, and $n^4 - 1$ is divisible by 3.

If n is not a multiple of 2 then n is odd, $n+1$ and $n-1$ are two consecutive even numbers, so one of them is also a multiple of 4, and $n^2 - 1 = (n+1)(n-1)$ is a multiple of 8.

On the other hand $n^2 + 1$ is even.

Consequently, $n^4 - 1 = (n^2+1)(n^2-1)$ is a multiple of $2 \cdot 8 = 16$.

Combined with the fact that $n^4 - 1$ is divisible by 3 we get that $n^4 - 1$ is divisible by 48, QED. $\square$

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Problem A3. We define a sequence $\{a_n\}_{n=1,2,3,\dots}$ recursively in the following way: $a_1 = 1$, $a_{n+1} = 2(a_n + 1)$ for $n = 1, 2, 3, \dots$. Find a close form for a_n.

- *Answer:* We define a new sequence $b_n = a_n + 2$, which verifies $b_1 = a_1 + 2 = 3$, $b_{n+1} = a_{n+1} + 2 = 2(a_n + 1) + 2 = 2a_n + 4 = 2(a_n + 2) = 2b_n$. Hence $b_n = 3 \cdot 2^{n-1}$, and

$$\boxed{a_n = 3 \cdot 2^{n-1} - 2}.$$

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Problem A4. Find a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \not\equiv 0$ and $f(4x) = f(2x) + f(x)$ for every real x.

- *Answer:* We try a function of the form $f(x) = Ax^\alpha$, so it must verify $A(4x)^\alpha = A(2x)^\alpha + Ax^\alpha$, or equivalently $4^\alpha = 2^\alpha + 1$. From here we get $2^\alpha = \frac{1\pm\sqrt{5}}{2}$. Only the positive solution makes sense in this case, and we get $\alpha = \log_2 \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Hence an answer is $f(x) = Ax^{\log_2 \phi}$ $(A \neq 0)$, for $x > 0$.

Since the exponent is not an integer we must be careful about how to extend f to negative x. The following extension works for every $x \in \mathbb{R}$:

$$f(x) = \begin{cases} Ax^{\log_2 \phi} & (x > 0) \\ 0 & (x = 0) \\ B|x|^{\log_2 \phi} & (x < 0) \end{cases}$$

where A, B do not need to be the same, but they should not be both zero if we want $f \not\equiv 0$.

(Note 1: The same solution can be expressed in other ways using the identity $x^{\log_2 \phi} = \phi^{\log_2 x}$.)

(Note 2: There are other possible solutions, e.g. $f(x) = F_n$ for $x \in [2^n, 2^{n+1})$ $(n \in \mathbb{Z})$, where F_n is the bidirectional Fibonacci sequence, and $f(-x) = f(x)$.)

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Problem A5. Let A, B, C, D be the following matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Is it possible to obtain the following matrix:

$$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

by multiplying the given matrices A, B, C, D (in any order and any number of times)?

- *Answer:* The answer is no. If P and Q are two square matrices of the same dimension then $\det(PQ) = \det P \det Q$. We have $\det A = \det B = \det C = \det D = 1$, so any product of those matrices will have determinant 1. But $\det E = -1$. $\square$

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Problem A6. Let x, y, z be three real numbers such that $0 < x, y < \pi$, $z = (x + y)/2$. Prove:

$$\sqrt{\frac{\sin x}{x}\frac{\sin y}{y}} \leq \frac{\sin z}{z}\,.$$

- *Answer:* Let $f : (0, \pi) \to \mathbb{R}$ the function $f(x) = \ln\left(\frac{\sin x}{x}\right) = \ln\sin x - \ln x$. We will prove $f(\frac{x+y}{2}) \geq \frac{1}{2}(f(x) + f(y))$.

The derivative of f is $f'(x) = \cot x - \frac{1}{x}$, and the second derivative is $f''(x) = -\frac{1}{\sin^2 x} + \frac{1}{x^2}$. Since $x > \sin x$ for $x \in (0, \pi)$ we have $f''(x) < 0$ for every $x \in (0, \pi)$, hence f is a concave function. As such it verifies $\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$ for $x, y \in (0, \pi)$ and $0 \leq \lambda \leq 1$. The result is the particular case $\lambda = 1/2$.

Finally, we have:

$$\sqrt{\frac{\sin x}{x}\frac{\sin y}{y}} = \sqrt{e^{f(x)}e^{f(y)}} = e^{\frac{1}{2}(f(x)+f(y))} \leq e^{f(\frac{x+y}{2})} = e^{f(z)} = \frac{\sin z}{z}\,.$$

$\square$